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On the symmetry of excitations in SU(2) Bethe Ansatz systems

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Abstract

Using the XXX Heisenberg chain as an example, based on the symmetry properties of the eigenstates with respect to reversing all the spins we argue, that the basic SU(2) symmetry of the model is inherited by the excitations with slight modifications only.

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Recently several attempts have been made to define regularized field theoretical models as some limits of certain completly integrable lattice models [1–4]. In these works the limiting models are identified through the structure and symmetry properties of the spectrum and the eigenstates. In some of the cases an SU(2) symmetry is present already in the initial lattice model [2,3], but in some this symmetry is recognized through degenerations developing in the limiting process only [3,4]. For this latter cases it can be instructive to see models, in which the connection between the basic SU(2) symmetry of the model and the symmetry of the excitations can be dirrectly detected. Our aim in the present note is to see this connection in the case of the isotropic Heisenberg chain, which is the symplest integrable SU(2) model.

The XXX Heisenberg chain is given by the Hamiltonian

$$H = \sum_{j=1}^{N} \vec{S}_j \vec{S}_{j+1} = \sum_{j=1}^{N} \left(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + S_j^z S_{j+1}^z \right) . \tag{1}$$

The Hilbert space is the tensor product of N spaces furnishing the doublet representation of SU(2), $S_j^{x,y,z}$ are the spin operators acting on the jth site, the $(N{+}1)$th site is identified with the first one, and we suppose N = even. The Hamiltonian commutes with all the components of the total spin

$$S^{x,y,z} = \sum_j S_j^{x,y,z} , \tag{2}$$

thus the eigenstates can be labelled by the values of the S^z and S^2, and the energy should show degenerations corresponding to SU(2) multiplets. Another operator commuting with the Hamiltonian and being important in our reasoning is

$$\hat{\Sigma} = \prod_{j=1}^{N} \sigma_j^x , \tag{3}$$

which represents a reflection on the x-axis, but in a basis given by the products of the S^z eigenstates of the individual spins $\hat{\Sigma}$ symply flips all the spins (the up ones down and the down ones up). The $S^z = 0$ eigenstates of (1) are expected to be eigenstates of this operation: otherwise certain points of the spectrum were twofold degenerated. This actually can happen accidentally, but is not forced by the symmetry, as $\hat{\Sigma}$ has one-dimensional representations only. In case of normal SU(2) the eigenvalue of $\hat{\Sigma}$ can be given as

$$\Sigma = (-1)^{N/2-l} \tag{4}$$

with l defined by the spin-length $S^2 = l(l+1)$. This is a consequence of the fact, that any $S^2 = l(l+1)$, $S^z = 0$ state can be given as linear combination of states built up as products of l triplet and $N/2 - l$ singlet pairs [5].

The Hamiltonian (1) has been diagonalized by Bethe Ansatz (BA) [6–9], and the BA equations (BAE) were analyzed [10–13]. Due to these studies the structure of the low energy states and the spectrum are well known by now. The ground state is an SU(2) singlet, and as there is no parameter in it, it can be considered as a vacuum. The excited states can be described as scattering states of dressed particles (spinwaves). Each particle has a parameter usually called rapidity through which its energy and momentum can be given. The total energy and momentum of a state are given by the sums of the contributions of the vacuum and the individual particles. The momenta of the n particles are quantized through a set of $n+r$ equations of the BA type (higher level Bethe Ansatz equations). These equations determine in addition to the particle momenta a set of $r(\leq n/2)$ variables, which do not enter either into the energy, or the momentum, but are needed to give the spin of the state:

$$S^z = l\,, \quad \text{and} \quad S^2 = l(l+1) \quad \text{with} \quad l = \frac{n}{2} - r\,, \tag{5}$$

These formulae and the fact, that the number of particles n and the number of sites in the chain (the number of real spins) N must be of the same parity suggest, that the spin of the individual particles is of length 1/2 [14].

It is remarkable, that the BA solutions give the $S^2 = S^z(S^z+1)$ ((5)), and the $S^z < l$ states should be constructed by the (repeted) application of the σ^- operator ($\sigma^\pm = S^x \pm i S^y$). The states form normal SU(2) multiplets (of the N spins forming thye chain), and it was very tempting to consider them as normal SU(2) multiplets of the dressed particles. This concept, however leads to a contradiction with the eigenvalue of $\hat{\Sigma}$: as the elements of a multiplet are $S^2 = l(l+1)$ eigenstates of the N spins forming the chain, the $S^z = 0$ member is symmetric or antisymmetric under $\hat{\Sigma}$ according to the eigenvalue (4), while in case of particles obeying normal SU(2), this eigenvalue should be $\Sigma = (-1)^{n/2-l}$, i.e. instead of the chainlength, the number of dressed particles should appear. Actually we think that Σ factorizes as

$$\Sigma = \Sigma_{vac.}\Sigma_{part.} \tag{6}$$

with $\Sigma_{vac.}$ and $\Sigma_{part.}$ being the eigenvalues corresponding to the vacuum and to the symmetry of the spin configuration of the particles, respectively:

$$\Sigma_{vac.} = (-1)^{N/2}\,, \quad \text{and} \quad \Sigma_{part.} = (-1)^l\,. \tag{7}$$

(We think this is the case as if $\Sigma_{part.}$ can be defined, it should be independent of the chainlength.) We should emphasize, there is no contradiction between the spin-length and the spinreversal symmetry, as long as we think in terms of N spins, the contradiction arise if we want to interpret them as the spinlength and the symmetry of the n spins caried by the dressed particles.

Now we show, that the above contradiction can be dissolved supposing the particles obey a modified SU(2) equivalent to a q-deformed SU(2) at $q = -1$. Actually we show the structure

in which the $S^z = 0$ member of a $2l+1$-fold degenerated multiplet is the eigenfunction of spin-reversal with the eigenvalue $\Sigma_{part.}$ of (7).

Let us define the σ operators of the n (n=even) spins as

$$\sigma^z = \sum_{j=1}^{n} \sigma_j^z \,, \quad \sigma^+ = p \sum_{j=1}^{n} (-1)^{j-1} \sigma_j^+ \,, \quad \sigma^- = \sum_{j=1}^{n} (-1)^{j-1} \sigma_j^- \,, \tag{8}$$

with p being ± 1 and σ_j acting on the spin of the jth particle. In the case $p = 1$ these operators obey the normal SU(2) commutation relations, but correspond to an unusual comultiplication:

$$\left[\sigma^+, \sigma^-\right] = \sigma^z \,, \qquad \left[\sigma^z, \sigma^\pm\right] = \pm 2\sigma^\pm \,, \tag{9}$$

$$\Delta(\sigma^z) = \sigma^z \otimes 1 + 1 \otimes \sigma^z \,, \qquad \Delta(\sigma^\pm) = \sigma^\pm \otimes 1 + (-1)^{\sigma^z} \otimes \sigma^\pm \,. \tag{10}$$

For $p = -1$ (8) correspond to a q-deformed SU(2) at $q = -1$ where the commutation relations an the cooproducts are

$$\left[\sigma^+, \sigma^-\right] = (-1)^{\sigma^z - 1} \sigma^z \,, \qquad \left[\sigma^z, \sigma^\pm\right] = \pm 2\sigma^\pm \,, \tag{11}$$

$$\begin{aligned} \Delta(\sigma^z) &= \sigma^z \otimes 1 + 1 \otimes \sigma^z \,, \\ \Delta(\sigma^+) &= \sigma^+ \otimes (-1)^{\sigma^z} + 1 \otimes \sigma^+ \,, \\ \Delta(\sigma^-) &= \sigma^- \otimes 1 + (-1)^{\sigma^z} \otimes \sigma^- \,. \end{aligned} \tag{12}$$

The two choices are equivalent, as redefining σ^+ as

$$\sigma^+ \to (-1)^{\sigma^z - 1} \sigma^+ \tag{13}$$

interchanges the $p = 1$ and $p = –1$ cases (n = even). In the folloving we take $p = 1$. The (8) operators can be obtained from the ‘normal’ SU(2) σs

$$\sigma^z_{SU(2)} = \sum_{j=1}^{n} \sigma_j^z \,, \quad \sigma^\pm_{SU(2)} = \sum_{j=1}^{n} \sigma_j^\pm \,. \tag{14}$$

by rotating evry second spin by π around the z axis, i.e.:

$$\sigma^z = \sigma^z_{SU(2)} \,, \quad \sigma^\pm = \hat{O} \hat{\sigma}^\pm_{SU(2)} \hat{O}^{-1} \,, \tag{15}$$

with

$$\hat{O} = \exp\left\{ i\pi \sum (j-1)\sigma_j^z/2 \right\} . \tag{16}$$

As a consequence, states in the two Hilbert spaces can be connected so, that the corresponding states have the same eigenvalues for S^z and S^2 defined in their own Hilbert space ($S^z = \sigma^z/2$, $S^2 = \sigma^-\sigma^+ + S^z(S^z+1)$ resp. $S^z = \sigma^z_{SU(2)}/2$, $S^2 = \sigma^-_{SU(2)}\sigma^+_{SU(2)} + S^z_{SU(2)}(S^z_{SU(2)}+1)$):

$$|\phi\rangle = \hat{O}|\phi\rangle_{SU(2)} . \tag{17}$$

As, however,

$$\hat{O}^{-1}\hat{\Sigma}\hat{O} = (-1)^{n/2}\hat{\Sigma} , \qquad \left(\text{now} \quad \hat{\Sigma} = \prod_{j=1}^{n} \sigma_j^x \right) \tag{18}$$

the $S^z = 0$ states have diferent eigenvalues under the spin reversal $\hat{\Sigma}$ (one corresponding to (7), the other being analogous to (4)):

$$\hat{\Sigma}|\phi\rangle = (-1)^l|\phi\rangle \quad \text{and} \quad \hat{\Sigma}|\phi\rangle_{SU(2)} = (-1)^{n/2-l}|\phi\rangle_{SU(2)} . \tag{19}$$

(In symple terms: among the two particle states in the (8) structure contrary to the case of normal SU(2) the state $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ is the $S^z = 0$ member of the triplet, and the singlet is $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$.)

As a summary, based on the above we may say, that even though we can not construct the spin operators of the dressed particles, having $(2l+1)$-fold degenerated multiplets where within a multiplet the states are distinguished by a quantum number S^z ($l \geq S^z = \text{integer} \geq -l$), the eigenvalue $\Sigma = (-1)^l$ indicates that the spin space of the particles is of the above structure.

Finally we note, that we expect to be general, that in BA systens, where the excited states can be described in terms of dressed particles, the underlying SU(2) symmetry of the system appears modified in the excitations: the contradiction to be dissolved between the Σ for N particle building up the system and the Σ for n dressed particles (namely that the first do not depend on n) is manifest in other systems too regardless of the actual form of the Hamiltonian.

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